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Factorization Method and Special Orthogonal Functions

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Abstract We present a general construction for ladder operators for the special orthogonal functions based on Nikiforov-Uvarov mathematical formalism. A list of creation and annihilation operators are provided for the well known special functions. Furthermore, we establish the dynamic group associated with these operators.

Keywords Factorization method · Nikiforov-Uvarov method · Special function

1 Introduction

Factorization method is a powerful tool for solving second order differential equations and a operator language formalism for consideration of spectral problems [1–5]. This method has been historically attributed to the pioneering works of Schrodinger in quantum mechanics [6], and developed by many authors like Infeld and Hull to different fields [7]. This approach is a kind of basic technique that reduces the dynamic equation of a given system into a simple one that is easier to handle. Its underlying idea is to consider a pair of linear ladder operators which can be obtained from a given second-order differential equation with boundary conditions. In this approach, exact solutions can be obtained once the second order differential equations are factorized by means of the linear ladder operators [8, 9]. In addition, this approach allows us to consider the hidden symmetry of the system through constructing a suitable Lie algebra, which can be realized by the ladder operators [10, 11]. In the theory of orthogonal polynomials, special functions are defined in this formalism as functions associated with similarity reductions of the factorization chains [12]. Factorization method is also useful in studying recursion relations associated with the special functions of hypergeometric type. On the other hand, a mathematical method developed by Nikiforov

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and Uvarov (NU) which makes it possible to present the theory of special functions by starting from a differential equation [13]. Recently, NU method has been used to solve equations in the relativistic and non-relativistic quantum mechanics, for some well known potentials [14–20].

The paper is organized as follows. In Sect. 2 we briefly review the NU mathematical formalism. In Sect. 3, the general form of some ladder operators of the special functions are obtained using associated recursion relations. We establish the dynamic group associated with the ladder operators and list them for some well known special functions, in Sect. 4. Section 5 is devoted to conclusion.

2 Preliminary Investigation of NU Method

NU method is based on reducing a second order linear differential equation into a generalized equation of hyper-geometric type. This method provides exact solutions in terms of special orthogonal functions as well as corresponding eigenvalues. In both relativistic and non relativistic quantum mechanics, the equation with a given real or complex potential can be solved by this method. Here, we use this method for solving KG equation with equal scalar and vector potentials. By introducing an appropriate coordinate transformation, one can rewrite this equation in the following form

$$\psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0, \tag{1}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials of degree two at most, and $\tilde{\tau}(s)$ is a polynomial of degree one at most. Now, if one takes the following factorization

$$\psi_n(s) = \phi(s) y_n(s). \tag{2}$$

Equation (1) reduces into a hyper-geometric type equation of the form

$$\sigma(s) y_n''(s) + \tau(s) y_n'(s) + \lambda y_n(s) = 0, \tag{3}$$

where $\tau(s) = \tilde{\tau}(s) + 2\pi(s)$ satisfy the condition $\tau'(s) < 0$, and $\pi(s)$ is defined by

$$\pi(s) = ((\sigma'(s) - \tilde{\tau}(s))/2) \pm \sqrt{(\sigma'(s) - \tilde{\tau}(s))^2/4 - \tilde{\sigma}(s) + \kappa\sigma(s)}, \tag{4}$$

here κ is a parameter. Determining of κ is the essential point in the calculation of $\pi(s)$. It is simply defined by setting the discriminant of the square root equal to zero. Therefore, one gets a general quadratic equation for κ. The values of κ can be used for calculation of energy eigenvalue by using

$$\lambda = \kappa + \pi'(s) = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s). \tag{5}$$

Polynomial solutions $y_n(s)$ are given by Rodrigues relation

$$y_n(s) = \frac{B_n}{\rho(s)}\left(\frac{d}{ds}\right)^n [\sigma^n(s)\rho(s)] \tag{6}$$

in which B_n is a normalization constant and $\rho(s)$ is the weight function satisfying the following condition

$$(\sigma\rho)' = \tau\rho. \tag{7}$$

By assuming that $\rho(s)$ is an analytic function on the closed contour C and its interior which surrounds the point $s = z$ and making use of the Cauchy's integral theorem, one can write an integral representation for Rodrigues relation (6) as follows

$$y_n(s) = \frac{C_n}{\rho(s)} \oint_C \frac{\sigma^n(z)\rho(z)}{(z-s)^{n+1}} dz, \tag{8}$$

where $C_n = \frac{n!B_n}{2\pi i}$. The integral representation of hypergeometric-type functions provides a convenient way to derive recursion relations. Two recursion relations which play an important role in the theory of orthogonal polynomials are as follows [21]

$$\begin{aligned} &(\alpha_n s + \beta_n) y_n - \sigma(s) y_n' + \gamma_n y_{n+1} = 0, \\ &(\tilde{\alpha}_n s + \tilde{\beta}_n) y_n - \sigma(s) y_n' + \tilde{\gamma}_n y_{n-1} = 0, \end{aligned} \tag{9}$$

where, the coefficients are given by

$$\begin{aligned} &\alpha_n = -\tau'_{(n-1)/2}(s), \quad \beta_n = -\frac{\tau_n(0)\tau'_{(n-1)/2}(s)}{\tau'_n(s)}, \quad \gamma_n = (n+1)\frac{C_n \tau'_{(n-1)/2}(s)}{C_{n+1}\tau'_n(s)}, \\ &\tilde{\alpha}_n = \frac{n}{2}\sigma''(s), \quad \tilde{\beta}_n = n\left[\sigma'(0) - \frac{\sigma''(s)\tau_{n-1}(0)}{2\tau'_{n-1}(s)}\right], \\ &\tilde{\gamma}_n = \frac{C_n}{C_{n-1}}\left\{\sigma(0)\tau'_{n-1}(s) - \tau_{n-1}(0)\left[\sigma'(0) - \frac{\sigma''(s)\tau_{n-1}(0)}{2\tau'_{n-1}(s)}\right]\right\}, \end{aligned} \tag{10}$$

in which

$$\tau_n(s) = \tau(s) + n\sigma'(s).$$

Moreover, the function $\phi(s)$ satisfies the condition

$$\phi'(s)/\phi(s) = \pi(s)/\sigma(s). \tag{11}$$

3 Ladder Operators

In this section we derive creation and annihilation operators for the special orthogonal functions based on NU formalism. Let us look for a first order differential operators of the following form

$$\hat{M}_\pm = A(s)\frac{d}{ds} + B(s) \tag{12}$$

with constraint

$$\hat{M}_\pm \psi_n(s) = m_\pm \psi_{n\pm 1}(s). \tag{13}$$

By applying $\frac{d}{ds}$ on the wave function $\psi_n(s)$ defined by (2) we get

$$\begin{aligned}\frac{d}{ds}\psi_n(s) &= \frac{\pi}{\sigma}\phi y_n + \phi\frac{d}{ds}y_n \\ &= \sigma^{-1}\{(\pi + \alpha_n s + \beta_n)\phi y_n + \gamma_n \phi y_{n+1}\}\end{aligned} \tag{14}$$

where condition (7) and recursion relation (9) have been used, respectively. Rearranging this expression and using (2) leads to

$$\left\{\sigma\frac{d}{ds} - (\pi + \alpha_n s + \beta_n)\right\}\psi_n(s) = \gamma_n \psi_{n+1}(s). \tag{15}$$

By comparing this relation with (13) we get

$$\hat{M}_+ = -(\pi + \alpha_n s + \beta_n) + \sigma\frac{d}{ds}, \quad m_+ = \gamma_n. \tag{16}$$

Similar procedure leads to the following expression for annihilation operator

$$\hat{M}_- = -(\pi + \tilde{\alpha}_n s + \tilde{\beta}_n) + \sigma\frac{d}{ds}, \quad m_- = \tilde{\gamma}_n. \tag{17}$$

4 Realization of Dynamic Group

To establish the dynamic group associated with the ladder operators $\hat{M}_\pm$, we calculate the commutator $[\hat{M}_+, \hat{M}_-]$

$$[\hat{M}_+, \hat{M}_-]\psi_n(s) = 2m_0\psi_n(s), \tag{18}$$

where we have introduced the eigenvalue

$$m_0 = \frac{1}{2}(\tilde{\gamma}_n\gamma_{n-1} - \tilde{\gamma}_{n+1}\gamma_n).$$

As a consequence, we can define the operator

$$\hat{M}_0 = \frac{1}{2}(\tilde{\gamma}_{\hat{n}}\gamma_{\hat{n}-1} - \tilde{\gamma}_{\hat{n}+1}\gamma_{\hat{n}}) \tag{19}$$

in which $\hat{n}$ is the number operator. $\hat{M}_0$ satisfies the following relations

$$[\hat{M}_+, \hat{M}_-] = 2\hat{M}_0 \qquad [\hat{M}_0, \hat{M}_\pm] = \pm\hat{M}_\pm \tag{20}$$

which can be recognized as commutation relation of the generators of a non-compact Lie algebra SU(1,1). The corresponding Casimir operator is defined by

$$\hat{C} = \hat{M}_0(\hat{M}_0 - 1) - \hat{M}_+\hat{M}_-.$$

Now, as an example of application of the factorization method, let us consider the Hermite differential equation

$$H_n''(s) - 2sH_n'(s) + 2nH_n(s) = 0. \tag{21}$$

Table 1 Some ladder operators associated with the known special functions

y_n	$P_n^{(\alpha,\beta)}$	$P_n(s)$	$L_n^{(\alpha)}(s)$	$H_n(s)$	$J_n(s)$
$\tilde{\tau}(s)$	s	$-2s$	$\alpha+1-s$	$-2s$	s
$\sigma(s)$	$1-s^2$	$1-s^2$	s	1	s^2
$\tilde{\sigma}(s)$	$(1-s)^{\alpha+1}(1+s)^{\beta+1}$	$n(n+1)(1-s^2)$	ns	$2n$	s^2-n^2
$\hat{M}_+$	$-(n+\alpha+\beta+1)s + (1-s^2)\frac{d}{ds}$	$-(n+1)s + (1-s^2)\frac{d}{ds}$	$(n+\alpha+1-s) + s\frac{d}{ds}$	$-2s+\frac{d}{ds}$	$n+1/2+s^2\frac{d}{ds}$
$\hat{M}_-$	$(n+\alpha+\beta)s + (1-s^2)\frac{d}{ds}$	$ns+(1-s^2)\frac{d}{ds}$	$-n+s\frac{d}{ds}$	$\frac{d}{ds}$	$-(n+1/2)+s^2\frac{d}{ds}$

Comparing this equation with (1), leads to the following functions

$$\tilde{\tau}(s) = -2s, \quad \sigma(s) = 1, \quad \tilde{\sigma}(s) = 2n. \tag{22}$$

By substituting them into relation (10) the corresponding coefficients α_n, β_n, $\tilde{\alpha}_n$, $\tilde{\beta}_n$ are obtained and lead to the following operators

$$\hat{M}_+ = -2s + \frac{d}{ds}, \quad \hat{M}_- = \frac{d}{ds}, \quad \hat{M}_0 = \mathbf{1}. \tag{23}$$

By the similar procedure of obtaining these operators, one can find corresponding operators of generalized differential equation of hyper-geometric type. Some of the ladder operators associated with the well known special functions are listed in Table 1.

These results are entirely consistent with those presented using several approaches like the supersymmetric quantum mechanics [22, 23] and the group theory [24–27].

5 Conclusion

We have derived the creation and annihilation operators for the special orthogonal functions by using the factorization method, based on Nikiforov-Uvarov formalism. It was found that these operators satisfy the commutation relations of the generators of the dynamic group SU(1, 1). A table of some ladder operators for the well known special functions has been listed.

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